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INDEPENDENT-STRESS METHOD FOR ANALYSIS OF NONLINEAR ANTIPLANE STRAIN

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The stress field in a cylindrical body under antiplane strains is studied using the nonlinear theory of elasticity in actual variables under assumptions of the absence of body forces and weak nonlinearity of the elastic potential. The stresses are determined by solving the nonlinear boundary-value problem for two independent stresses in polar coordinates of the physical and stress planes. Analytical solutions of the nonlinear problems are obtained. The effect of potential nonlinearity is studied. It is shown that the nonlinear problem can be solved using the harmonic-equation solution corresponding to the linear potential.

Key words: *displacement, strain, stress, potential, nonlinearity, analytical solution, stress plane, boundary-value problem.*

1. We consider nonlinear antiplane strain of a cylindrical elastic body. In the actual variables x_1, x_2, and x_3 ($x_1 = x$ and $x_2 = y$ are the transverse coordinates and $x_3 = z$ is the longitudinal coordinate), the transverse displacements vanish and the longitudinal displacement does not depend on the z coordinate: $u_x = u_y = 0$ and $u_z = w(x, y)$ [1]. In this case, the body behaves like an incompressible body. The invariants E_k of the Almansi strains A_{kl} are nonpositive and can be expressed in terms of the linear invariant E_1. For an isotropic body, the elastic potential $U = U(E_1)$ is also a function of the linear invariant.

By virtue of the strain-compatibility equations

$$2E_{11} = -(2E_{31})^2, \quad 2E_{22} = -(2E_{32})^2, \quad 2E_{33} = 0, \quad 2E_{12} = -2E_{31}2E_{32}, \quad \frac{\partial E_{32}}{\partial x} = \frac{\partial E_{31}}{\partial y}$$

and the inverted Murnaghan law [2] $2E_{kl} = -(P_{kl} + q\delta_{kl})/U'$ (q is the pressure, δ_{kl} is the Kronecker symbol, and $U' = dU/dE_1$), the Cauchy stresses P_{kl} are related to the pressure and independent stresses P_{zx} and P_{zy} by the nonlinear dependences (here and below, numerical subscripts are replaced by literal subscripts)

$$P_{xx} = -q + P_{zx}^2/U', \quad P_{yy} = -q + P_{zy}^2/U', \quad P_{zz} = -q, \quad P_{xy} = P_{zx}P_{zy}/U', \tag{1}$$

and the quantities q, P_{zx}, and P_{zy} (independent of the longitudinal coordinate) are determined from the equations of equilibrium (in the absence of body forces) and differential stress-compatibility equation [3].

It is assumed that the derivative of the potential in relations (1) is expressed in terms of stresses. Eliminating the strain invariant from the relations $2E_1 = -(P_{zx}^2 + P_{zy}^2)/U'^2$ and $U' = N(2E_1)$, we obtain the expression in the implicit form

$$U' = N(-R^2/U'^2), \qquad R^2 = P_{zx}^2 + P_{zy}^2. \tag{2}$$

Integrating two equilibrium equations $\partial P_{1k}/\partial x_k = 0$ and $\partial P_{2k}/\partial x_k = 0$ transformed with the use of the third equation, we express pressure in terms of the elastic potential

$$q = h - U, \qquad h = \text{const}. \tag{3}$$

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The constant h is determined by the potential and end load and is equal to the average value of the potential in the cross section S of the body if the axial stress resultant vanishes:

$$h = \frac{1}{S} \int\limits_S U\, dS. \tag{4}$$

Independent stresses are determined from the nonlinear boundary-value problem for the differential stress-compatibility equation and equilibrium equation with the force conditions at the cross-sectional contour L of the body:

$$\frac{\partial}{\partial x} \frac{P_{zy}}{U'} - \frac{\partial}{\partial y} \frac{P_{zx}}{U'} = 0, \qquad \frac{\partial P_{zx}}{\partial x} + \frac{\partial P_{zy}}{\partial y} = 0, \tag{5}$$

$$P_{zx} = g_n n_x - g_t n_y, \qquad P_{zy} = g_n n_y + g_t n_x L \qquad \text{on} \quad L.$$

Here n_x and n_y are the components of the external normal to the cross section, U' is the solution of Eq. (2), and g_n and g_t are the stresses determined via the tangential p_t, normal p_n, and binormal p_b contour loads from the equations given in [3]:

$$p_t = g_n g_t / U', \quad p_n = U - h + g_n^2/U', \quad p_b = g_n \quad (R^2 = g_n^2 + g_t^2). \tag{6}$$

We consider the Rivlin–Saunders quadratic elastic potential U generalizing the Mooney linear potential U_0

$$U(E_1) = aE_1^2 - 2bE_1, \quad U_0(E_1) = -2bE_1 \quad (a > 0,\ b > 0,\ E_1 < 0), \tag{7}$$

which adequately describes large elastic strains of rubber-like materials [1]. The potential and its derivative can be expressed in terms of stresses. In this case, these quantities are related and dependence (2) is the cubic equation

$$U = (U'^2 - 4b^2)/(4a), \qquad U'^3 + 2bU'^2 + aR^2 = 0. \tag{8}$$

We write Eqs. (5) in an expanded form

$$2aP_{zx}P_{zy}\Big(\frac{\partial P_{zx}}{\partial x} - \frac{\partial P_{zy}}{\partial y}\Big) + (U'^3 - 2aP_{zx}^2)\,\frac{\partial P_{zy}}{\partial x} - (U'^3 - 2aP_{zy}^2)\,\frac{\partial P_{zx}}{\partial y} = 0,$$
$$\frac{\partial P_{zx}}{\partial x} + \frac{\partial P_{zy}}{\partial y} = 0 \tag{9}$$

and put them into correspondence to the second-order characteristic matrix D_{kl} with the following components and determinant [4]:

$$D_{xx} = 2aP_{zx}P_{zy}v_x - (U'^3 - 2aP_{zy}^2)v_y, \qquad D_{yy} = v_y,$$
$$D_{xy} = (U'^3 - 2aP_{zx}^2)v_x - 2aP_{zx}P_{zy}v_y, \qquad D_{yx} = v_x, \tag{10}$$
$$D = \det D_{kl} = D_{xx}D_{yy} - D_{xy}D_{yx} = -U'^3(v_x^2 + v_y^2) + 2a(P_{zx}v_x + P_{zy}v_y)^2.$$

The cubic equation in (8) has one real root (U_1') and two complex-conjugate roots (U_2', U_3') [5]. By virtue of the properties of the roots, the real root is negative:

$$aR^2 = -U_1'U_2'U_3' = -U_1'|U_2'|^2, \qquad U_1' < 0.$$

Hence, determinant (10) corresponding to the real root is positive: $D > 0$. Therefore, the characteristic equation $D = 0$ has no real roots [4]. Thus, the Rivlin–Saunders quadratic potential (7) corresponds to the elliptic system (5), for which the boundary-value problem is well-posed.

We now study the weak nonlinearity of potential (7) for which the coefficient of the quadratic term is small compared to that of the linear term: $a/(2b) \ll 1$. It follows that the elastic coefficient is small too: $m = a/(8b^3) \ll 1$. Using the linear (in terms of m) approximation of the derivative $U' = U_0' + mU_1'$, we determine U' and then express U with allowance for Eqs. (8):

$$U_0' = -2b, \qquad U_1' = -2bR^2, \qquad U' = -2b(1 + mR^2), \qquad U = R^2(2 + mR^2)/(8b). \tag{11}$$

In this approximation, potentials (11) remain nonlinear in terms of stresses and system (9) preserves the elliptic type.

It follows from (5) and (6) that the relation $R^2 = g_n^2 + g_t^2 = p_b^2 + g_t^2$ holds at the cross-sectional contour. With allowance for this relation and the values of potentials (11), system (6) can be transformed. In the approximation considered, its first and third equations determine the boundary stresses g_t and g_n and the second equation determines the load constraint:

$$g_t = -(2bp_t/p_b^3)[p_b^2 + m(4b^2p_t^2 + p_b^4)], \qquad g_n = p_b,$$
$$(p_b^4 + 4b^2p_t^2)[2p_b^2 + m(3p_b^4 + 28b^2p_t^2 - 8bp_b^2(p_n + h))] = 4p_b^4(p_b^2 + 2b(p_n + h)). \tag{12}$$

Thus, nonlinear antiplane strain occurs in the body if the load elements and elastic coefficients are related.

2. In the relations obtained above, we change the Cartesian coordinates x, y, and z to the cylindrical coordinates r, v, and z: $x = r\cos v$, $y = r\sin v$, and $z = z$. In this case, the relation between the components of the normal and stresses in new variables is given by

$$n_x = n_r\cos v - n_v\sin v, \qquad n_y = n_r\sin v + n_v\cos v, \qquad n_z = n_z,$$

$$P_{xx} = P_{rr}\cos^2 v + P_{vv}\sin^2 v - P_{rv}\sin 2v, \qquad P_{yy} = P_{rr}\sin^2 v + P_{vv}\cos^2 v + P_{rv}\sin 2v,$$
$$P_{zz} = P_{zz}, \qquad P_{zx} = P_{zr}\cos v - P_{zv}\sin v, \qquad P_{zy} = P_{zr}\sin v + P_{zv}\cos v, \tag{13}$$
$$P_{xy} = (P_{rr} - P_{vv})\sin v\cos v + P_{rv}\cos 2v, \qquad R^2 = P_{zx}^2 + P_{zy}^2 = P_{zr}^2 + P_{zv}^2.$$

In accordance with (1) and (13), the cylindrical components of the stresses are expressed in terms of the quantities q, P_{zr}, and P_{zv}:

$$P_{rr} = -q + P_{zr}^2/U', \qquad P_{vv} = -q + P_{zv}^2/U', \qquad P_{zz} = -q, \qquad P_{rv} = P_{zr}P_{zv}/U'. \tag{14}$$

Here q, U, and U' are determined by formulas (3) and (11) and the independent stresses P_{zr} and P_{zv} are found from the boundary-value problem

$$U'\Big(\frac{\partial(rP_{zv})}{\partial r} - \frac{\partial P_{zr}}{\partial v}\Big) - rP_{zv}\frac{\partial U'}{\partial r} + P_{zr}\frac{\partial U'}{\partial v} = 0, \qquad \frac{\partial(rP_{zr})}{\partial r} + \frac{\partial P_{zv}}{\partial v} = 0,$$
$$P_{zr} = g_n n_r - g_t n_v, \qquad P_{zv} = g_n n_v + g_t n_r \qquad \text{on } L. \tag{15}$$

In the case where the stresses depend on one polar coordinate, Eqs. (15) admit simple analytical solutions. Let the stresses be functions of the polar radius. The equations become

$$U'\frac{d(rP_{zv})}{dr} - rP_{zv}\frac{dU'}{dr} = 0, \qquad \frac{d(rP_{zr})}{dr} = 0, \qquad U' = -2b(1 + m(P_{zr}^2 + P_{zv}^2)).$$

Integrating these equations, we obtain the relations

$$rP_{zr} = A, \qquad mBP_{zv}^2 - rP_{zv} + B(r^2 + mA^2)/r^2 = 0, \qquad A = \text{const}, \qquad B = \text{const},$$

which, with allowance for the condition $m \ll 1$, yield two solutions with free parameters A and B:

$$P_{zr} = \frac{A}{r}, \qquad P_{zv} = \frac{r}{mB} - \frac{B}{r}\Big(1 + m\,\frac{A^2 + B^2}{r^2}\Big); \tag{16}$$

$$P_{zr} = \frac{A}{r}, \qquad P_{zv} = \frac{B}{r}\Big(1 + m\,\frac{A^2 + B^2}{r^2}\Big). \tag{17}$$

If the stresses depend only on the polar angle, Eqs. (15) become

$$U'\Big(P_{zv} - \frac{dP_{zr}}{dv}\Big) - P_{zr}\frac{dU'}{dv} = 0, \qquad P_{zr} + \frac{dP_{zv}}{dv} = 0, \qquad U' = -2b(1 + m(P_{zr}^2 + P_{zv}^2)).$$

This system reduces to the equations

$$P_{zr} = -\frac{dP_{zv}}{dv}, \qquad \Big(\frac{d^2P_{zv}}{dv^2} + P_{zv}\Big)\Big[1 + mP_{zv}^2 - m\Big(\frac{dP_{zv}}{dv}\Big)^2\Big] = 0.$$

Integrating these equations, we obtain the following solutions, which depend on the parameters T, E, and f:

$$P_{zv} = T\cos v + E\sin v, \qquad P_{zr} = T\sin v - E\cos v, \qquad T = \text{const}, \qquad E = \text{const}; \tag{18}$$

$$P_{zv} = \sinh\,(f \pm v)/\sqrt{m}, \qquad P_{zr} = \mp\cosh\,(f \pm v)/\sqrt{m}, \qquad f = \text{const}. \tag{19}$$

Solutions (16) and (19) are valid only for the quadratic elastic potential. As $m \to 0$, the potential becomes linear and the corresponding solutions increase unlimitedly and, hence, become meaningless. Thus, some of the solutions of the nonlinear system (15) do not have linear analogs. We use the solutions obtained to solve some particular boundary-value problems and determine the corresponding pressure, dependent stresses, and load in the approximation considered.

Let the cross section of the body be the exterior of a circle of radius r_0 and the coordinate origin be located at the circle center. In this case, the components of the normal are $n_r = -1$ and $n_v = 0$, and the boundary conditions in (15) have the form

$$P_{zr} = -g_n, \qquad P_{zv} = -g_t \qquad \text{for} \quad r = r_0. \tag{20}$$

We use solution (17):

$$P_{zr} = \frac{A}{r}, \qquad P_{zv} = \frac{B}{r}\Big(1 + m\,\frac{A^2 + B^2}{r^2}\Big), \qquad R^2 = \frac{A^2 + B^2}{r^2}\Big(1 + m\,\frac{2B^2}{r^2}\Big). \tag{21}$$

For this solution, potentials (11) are functions of the polar radius:

$$U = \frac{A^2 + B^2}{4br^2}\Big(1 + m\,\frac{A^2 + 5B^2}{2r^2}\Big), \qquad U' = U_0'\Big(1 + m\,\frac{A^2 + B^2}{r^2}\Big). \tag{22}$$

The constant h in (4) is determined for the exterior of the circle by calculating the limit of the mean value of potential (22) in an annular ring that encloses the hole as the ring expands unlimitedly. The calculations show that this constant vanishes and pressure (3) differs from the elastic potential only in sign:

$$h = \lim_{r_* \to \infty} \frac{1}{r_*^2 - r_0^2} \int\limits_{r_0^2}^{r_*^2} \frac{A^2 + B^2}{4br^2}\Big(1 + m\,\frac{A^2 + 5B^2}{2r^2}\Big)\,dr^2 = 0,$$

$$q = -\frac{A^2 + B^2}{4br^2}\Big(1 + m\,\frac{A^2 + 5B^2}{2r^2}\Big). \tag{23}$$

Like the independent stresses and pressure (23), the dependent stresses (14) are functions of the radius:

$$P_{rr} = \frac{B^2 - A^2}{4br^2} + m\,\frac{5(A^2 + B^2)^2}{8br^4}, \qquad P_{vv} = \frac{A^2 - B^2}{4br^2} + m\,\frac{(A^2 + B^2)^2}{8br^4},$$

$$P_{zz} = \frac{A^2 + B^2}{4br^2}\Big(1 + m\,\frac{A^2 + 5B^2}{2r^2}\Big), \qquad P_{rv} = -\frac{AB}{2br^2}.$$

Stresses (21) vanish at infinity: $P_{zr}^{\infty} = P_{zv}^{\infty} = 0$, and the contour stresses (20) become constant:

$$g_n = -\frac{A}{r_0}, \qquad g_t = -\frac{B}{r_0}\Big(1 + m\,\frac{A^2 + B^2}{r_0^2}\Big) \qquad \text{for} \quad r = r_0. \tag{24}$$

It follows from (6) and (24) and the expression for potentials (22) that the loads also take constant values

$$p_t = -\frac{AB}{2br_0^2}, \qquad p_n = \frac{B^2 - A^2}{4br_0^2} + m\,\frac{5(A^2 + B^2)^2}{8br_0^4}, \qquad p_b = -\frac{A}{r_0}, \tag{25}$$

which satisfy constraints (12).

The normal stress P_{vv}^L at the site orthogonal to the contour and its value P_{vv}^{0L} for the linear ($m = 0$) potential are given by

$$P_{vv}^L = \frac{A^2 - B^2}{4br_0^2} + m\,\frac{(A^2 + B^2)^2}{8br_0^4}, \qquad P_{vv}^{0L} = \frac{A^2 - B^2}{4br_0^2}. \tag{26}$$

It follows from (26) that, for load (25), the contour of the hole is extended for $A^2 > B^2$ and compressed for $A^2 < B^2$; taking into account potential nonlinearity increases extension and decreases compression.

In the exterior of the circle $r = r_0$, solution (18)

$$P_{zr} = T\sin v - E\cos v, \qquad P_{zv} = T\cos v + E\sin v, \qquad R^2 = T^2 + E^2 = \text{const} \tag{27}$$

yields other stresses and loads. For stresses (27), potentials (11) are constants:

$$U' = U_0'[1 + m(T^2 + E^2)], \qquad U = (T^2 + E^2)[2 + m(T^2 + E^2)]/(8b).$$

In accordance with (4), the constant h is equal to the potential: $h = U$, which implies that pressure (3) vanishes: $q = 0$. Hence, the dependent stresses (14) along with stresses (27) in the exterior of the hole are functions of the angle:

$$P_{rr} = -\frac{(T\sin v - E\cos v)^2}{2b(1+m(T^2+E^2))}, \qquad P_{vv} = -\frac{(T\cos v + E\sin v)^2}{2b(1+m(T^2+E^2))},$$

$$P_{zz} = 0, \qquad P_{rv} = \frac{(E\cos v - T\sin v)(T\cos v + E\sin v)}{2b(1+m(T^2+E^2))}.$$

At infinity where the polar angle is indeterminate, stresses (27) are also indeterminate, whereas at the hole contour, they determine the contour stresses (20) in the form

$$g_n = -T\sin v + E\cos v, \qquad g_t = -T\cos v - E\sin v.$$

The components of the contour load (6) are functions of the angle:

$$p_t = \frac{(E\cos v - T\sin v)(T\cos v + E\sin v)}{2b(1+m(T^2+E^2))}, \qquad p_n = -\frac{(E\cos v - T\sin v)^2}{2b(1+m(T^2+E^2))},$$

$$p_b = E\cos v - T\sin v,$$

which agree with the load constraints (12).

At the sites orthogonal to the contour, the normal stresses P^L_{vv} and P^{0L}_{vv} (for the quadratic and linear potentials, respectively) are given by

$$P^L_{vv} = -\frac{(T\cos v + E\sin v)^2}{2b(1+m(E^2+T^2))}, \qquad P^{0L}_{vv} = -\frac{(T\cos v + E\sin v)^2}{2b}.$$

It follows that, for the load determined above, the hole contour is compressed and taking into account potential nonlinearity decreases compression.

3. Another method of studying problem (5) is based on using stress potentials, namely, displacement and stress function. The first of these potentials has a mechanical meaning. The first equation in system (5) is satisfied if the stresses (normalized to U') are expressed in terms of the axial displacement $w(x, y)$:

$$\frac{P_{zx}}{U'} = -\frac{\partial w}{\partial x}, \qquad \frac{P_{zy}}{U'} = -\frac{\partial w}{\partial y}. \tag{28}$$

The second equation is satisfied if the stresses are expressed in terms of the stress function $t(x, y)$:

$$P_{zx} = \frac{\partial t}{\partial y}, \qquad P_{zy} = -\frac{\partial t}{\partial x}. \tag{29}$$

Eliminating the stresses from (28) and (29), we obtain the nonlinear system of equations for the functions w and t:

$$\frac{\partial w}{\partial x} = -\frac{1}{U'}\frac{\partial t}{\partial y}, \qquad \frac{\partial w}{\partial y} = \frac{1}{U'}\frac{\partial t}{\partial x}, \qquad U' = U'(R^2), \qquad R^2 = \Big(\frac{\partial t}{\partial x}\Big)^2 + \Big(\frac{\partial t}{\partial y}\Big)^2. \tag{30}$$

Differentiating system (30), one can eliminate one potential and obtain a second-order differential equation for the other potential. Expressing the derivative of the elastic potential in terms of the strain invariant $U'(E_1)$ and taking into account the expression for the invariant [3] $2E_1 = -|\nabla w|^2$, we obtain the equation for the displacement:

$$\Big[U' - U''\Big(\frac{\partial w}{\partial x}\Big)^2\Big]\frac{\partial^2 w}{\partial x^2} - 2U''\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\,\partial y} + \Big[U' - U''\Big(\frac{\partial w}{\partial y}\Big)^2\Big]\frac{\partial^2 w}{\partial y^2} = 0. \tag{31}$$

In particular, for the linear elastic potential (7), the derivatives of the potential are constant: $U'_0 = -2b$ and $U''_0 = 0$, and Eq. (31) for the displacement (denoted by w_0) becomes the Laplace equation

$$\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial y^2} = 0. \tag{32}$$

Equations (30) can be written in the form of a linear system for the same functions if the independent variables are chosen appropriately. The equations are nonlinear due to the quantity U', which depends on the variable $R = \sqrt{P^2_{zx} + P^2_{zy}}$. Therefore, we pass from the Cartesian coordinates x and y of the physical plane to the polar coordinates R and V of the stress plane (P_{zx}, P_{zy}): $P_{zx} = R\cos V$ and $P_{zy} = R\sin V$. To this end, we consider the expression

$$dt + iU'\,dw = -i(P_{zx} - iP_{zy})(dx + i\,dy) = -i\,\mathrm{Re}^{-iV}\ dz \qquad (z = x + iy),$$

in which P_{zx} and P_{zy} are determined by formulas (28) and (29). Assuming that z, t, and w are functions of R and V, with allowance for $R \neq 0$, we obtain

$$dz = (i\mathrm{e}^{iV}/R)(dt + iU'\,dw); \tag{33}$$

$$\frac{\partial z}{\partial R} = \frac{i\mathrm{e}^{iV}}{R}\Big(\frac{\partial t}{\partial R} + iU'\,\frac{\partial w}{\partial R}\Big), \qquad \frac{\partial z}{\partial V} = \frac{i\mathrm{e}^{iV}}{R}\Big(\frac{\partial t}{\partial V} + iU'\,\frac{\partial w}{\partial V}\Big). \tag{34}$$

Elimination of the function $z(R, V)$ from Eqs. (34) by equating the partial derivatives $\partial^2 z/\partial R\,\partial V = \partial^2 z/\partial V\,\partial R$ yields the relation

$$\frac{\partial t}{\partial R} + iU'\,\frac{\partial w}{\partial R} = \frac{i}{R}\Big(\frac{\partial t}{\partial V} + iU'\,\frac{\partial w}{\partial V}\Big) + \frac{\partial U'}{\partial R}\,\frac{\partial w}{\partial V}.$$

Separating the real and imaginary parts, we obtain the linear system of equations

$$\frac{\partial t}{\partial V} = RU'\,\frac{\partial w}{\partial R}, \qquad \frac{\partial t}{\partial R} = R\,\frac{d}{dR}\Big(\frac{U'}{R}\Big)\frac{\partial w}{\partial V}. \tag{35}$$

Eliminating the stress function from system (35), we obtain the equation for the axial displacement $w(R, V)$

$$\frac{\partial}{\partial R}\Big(RU'\,\frac{\partial w}{\partial R}\Big) - R\,\frac{d}{dR}\Big(\frac{U'}{R}\Big)\frac{\partial^2 w}{\partial V^2} = 0. \tag{36}$$

For the linear elastic potential, this relation becomes

$$R_0\,\frac{\partial}{\partial R_0}\Big(R_0\,\frac{\partial w_0}{\partial R_0}\Big) + \frac{\partial^2 w_0}{\partial V^2} = 0. \tag{37}$$

Hereinafter, the subscript 0 denotes quantities corresponding to the linear elastic potential.

The solution of Eq. (36) $w = w(R, V)$ with the coordinate-transformation formulas $R = R(x, y)$ and $V = V(x, y)$, which follow from Eq. (33), determine the displacement and stresses in the physical plane:

$$w(x, y) = w(R(x, y), V(x, y)),$$

$$P_{zx}(x, y) = R(x, y)\cos V(x, y), \quad P_{zy}(x, y) = R(x, y)\sin V(x, y).$$

To establish a relation between the coordinates in (34), we replace the stress-function gradients by the displacement gradients using formulas (35). As a result, we obtain

$$F = \frac{\partial z}{\partial R} = \mathrm{e}^{iV}\Big(i\,\frac{d}{dR}\Big(\frac{U'}{R}\Big)\frac{\partial w}{\partial V} - \frac{U'}{R}\,\frac{\partial w}{\partial R}\Big), \quad G = \frac{\partial z}{\partial V} = \frac{U'}{R}\,\mathrm{e}^{iV}\Big(iR\,\frac{\partial w}{\partial R} - \frac{\partial w}{\partial V}\Big). \tag{38}$$

Integration of these equations according to [6] yields

$$z = \int F(R, V)\,dR + G(R, V)\,dV. \tag{39}$$

Since the functions F and G satisfy the condition $\partial F/\partial V = \partial G/\partial R$, the integrand in (39) is the total differential. Hence, for each solution of Eq. (36), the integral in (39) is independent of the integration path; separating the real and imaginary parts in equality (39), we obtain the transformation $x = x(R, V)$ and $y = y(R, V)$.

By virtue of (38), the determinant of the coordinate transformation is expressed in terms of the displacement and elastic potential:

$$\frac{\partial(x, y)}{\partial(R, V)} = \frac{\partial(x, y)}{\partial(z, \bar z)}\,\frac{\partial(z, \bar z)}{\partial(R, V)} = -\frac{U'^2}{R}\Big(\frac{\partial w}{\partial R}\Big)^2 + \frac{U'}{R}\,\frac{d}{dR}\Big(\frac{U'}{R}\Big)\Big(\frac{\partial w}{\partial V}\Big)^2.$$

In the approximation linear in m, the potential U' from (11) satisfies the equality

$$\frac{U'}{R}\,\frac{d}{dR}\Big(\frac{U'}{R}\Big) = -\frac{4b^2}{R^3}\,(1 + mR^2)(1 - mR^2) \approx -\frac{4b^2}{R^3},$$

which implies that the determinant is finite and nonzero (except for singular points at which $\partial w/\partial R = \partial w/\partial V = 0$ or $R = 0$):

$$\frac{\partial(x,y)}{\partial(R,V)} = -\frac{1}{R^3}\Big[R^2(1+mR^2)\Big(\frac{\partial w}{\partial R}\Big)^2 + \Big(\frac{\partial w}{\partial V}\Big)^2\Big] \neq 0,$$

It follows that there exists an inverse transformation $R = R(x,y)$ and $V = V(x,y)$.

For the elastic potential (11), the equation for displacement (36) is a linear second-order equation with variable coefficients

$$R^2(1+mR^2)\frac{\partial^2 w}{\partial R^2} + R(1+3mR^2)\frac{\partial w}{\partial R} + (1-mR^2)\frac{\partial^2 w}{\partial V^2} = 0. \qquad (40)$$

This equation admits simple analytical solutions. The displacement $w = cR^n\cos(kV)$, where $c = \text{const}$, is the solution of Eq. (40) if the parameters k and n satisfy the relation $n^2 - k^2 + mR^2(n^2+2n+k^2) = 0$. For an arbitrary elastic coefficient m, this equation is satisfied if $n^2 - k^2 = 0$ and $n^2 + 2n + k^2 = 0$, i.e., for $k = 1$ and $n = -1$, which yields the solution

$$w = (c/R)\cos V = (c/R)(\mathrm{e}^{iV} + \mathrm{e}^{-iV})/2. \qquad (41)$$

Solution (41) and the potential U' from (11) correspond to functions (38)

$$F = -2bc(\mathrm{e}^{2iV} + mR^2)/R^3, \qquad G = 2ibc(1+mR^2)\mathrm{e}^{2iV}/R^2,$$

for which the integral in (39) is taken in a finite form and equality (39) gives a transcendental relation between the coordinates of the physical and stress planes:

$$z = bc(-m\ln R^2 + (1+mR^2)\mathrm{e}^{2iV}/R^2). \qquad (42)$$

The complex equality (42) determines the stresses in the physical plane $R(x,y)$, $V(x,y)$ implicitly. For weak physical nonlinearity, one can obtain these relation in the explicit form. To this end, we linearize these functions with respect to m

$$R(x,y) = R^0(x,y) + mR^1(x,y), \qquad V(x,y) = V^0(x,y) + mV^1(x,y). \qquad (43)$$

Substituting (43) into (42), we obtain

$$z = \frac{bc}{R^{02}}\,\mathrm{e}^{2iV^0} - mbc\Big[\ln R^{02} - \mathrm{e}^{2iV^0}\Big(1 - \frac{2R^1}{R^{03}} + \frac{2iV^1}{R^{02}}\Big)\Big].$$

Comparing the coefficients of the like powers of the parameter on different sides of the equality, we obtain the complex equations for the quantities R^0, R^1, V^0, and V^1:

$$z = \frac{bc}{R^{02}}\,\mathrm{e}^{2iV^0}, \qquad \ln R^{02} - \mathrm{e}^{2iV^0}\Big[1 - \frac{2}{R^{02}}\Big(\frac{R^1}{R^0} - iV^1\Big)\Big] = 0.$$

These equations imply

$$R^{02} = bc/r, \qquad \tan 2V^0 = \tan v = y/x, \qquad r = \sqrt{x^2+y^2},$$
$$\frac{R^1}{R^0} = \frac{bc}{2r}\Big(1 - \frac{x}{r}\ln\frac{bc}{r}\Big), \qquad V^1 = -\frac{bcy}{2r^2}\ln\frac{bc}{r}. \qquad (44)$$

For the approximation considered, quantities (43) correspond to the Cartesian stress components

$$P_{zx} = (R^0 + mR^1)\cos(V^0 + mV^1) = R^0\cos V^0[1 + m(R^1/R^0 - V^1\tan V^0)],$$

$$P_{zy} = (R^0 + mR^1)\sin(V^0 + mV^1) = R^0\sin V^0[1 + m(R^1/R^0 + V^1\cot V^0)].$$

Taking into account the relations that follow from (44)

$$\tan V^0 = \frac{\sqrt{1+\tan^2 2V^0} - 1}{\tan 2V^0} = \frac{r-x}{y}, \quad \sin V^0 = \frac{r-x}{\sqrt{2r(r-x)}}, \quad \cos V^0 = \frac{y}{\sqrt{2r(r-x)}},$$

we write Cartesian stresses in the form

$$P_{zx} = \frac{\sqrt{bc}}{r\sqrt{2}}\,\frac{y}{\sqrt{r-x}}\Big[1 + \frac{mbc}{2r}\Big(1 + \frac{r-2x}{r}\ln\frac{bc}{r}\Big)\Big],$$
$$P_{zy} = \frac{\sqrt{bc}}{r\sqrt{2}}\,\frac{r-x}{\sqrt{r-x}}\Big[1 + \frac{mbc}{2r}\Big(1 - \frac{r+2x}{r}\ln\frac{bc}{r}\Big)\Big]. \qquad (45)$$

In the exterior of the circle of radius r_0, stresses (45) vanish at infinity and have the variable values

$$P^L_{zx} = \frac{\sqrt{bc}}{\sqrt{r_0}} \cos\frac{v}{2}\Big[1 + \frac{mbc}{2r_0}\Big(1 + (1 - 2\cos v)\ln\frac{bc}{r_0}\Big)\Big],$$
$$P^L_{zy} = \frac{\sqrt{bc}}{\sqrt{r_0}} \sin\frac{v}{2}\Big[1 + \frac{mbc}{2r_0}\Big(1 - (1 + 2\cos v)\ln\frac{bc}{r_0}\Big)\Big] \qquad (46)$$

at the boundary circumference L ($x = r_0\cos v$ and $y = r_0 \sin v$). Given these stresses, the boundary load is calculated using formulas (20) and (6).

In this case, using (11) and (44), we obtain the elastic potential and its derivative

$$U = \frac{R^{02}}{8b}\Big(1 + 2m\frac{R^1}{R^0}\Big)(2 + mR^{02}) = \frac{c}{4r} + \frac{mbc^2}{8r^2}\Big(3 - 2\cos v \ln\frac{bc}{r}\Big),$$
$$U' = -2b(1 + mR^{02}) = -2b(1 + mbc/r). \qquad (47)$$

It follows from Eq. (11) that $U^1 > 0$ in the expression for the elastic potential $U = U^0 + mU^1$. To satisfy this condition, one should impose the constraint on the constant c in expression (47):

$$3 - 2\ln(bc/r_0) > 0. \qquad (48)$$

The constant h in (4) calculated in the exterior of the circle by limiting passage vanishes:

$$h = \lim_{r_*\to\infty} \frac{1}{\pi(r_*^2 - r_0^2)} \int_{r_0}^{r_*} r\,dr \int_0^{2\pi} U(r,v)\,dv = \lim_{r_*\to\infty} \frac{c}{2}\Big(\frac{1}{r_* + r_0} + \frac{3mb}{2}\,\frac{\ln r_* - \ln r_0}{r_*^2 - r_0^2}\Big) = 0.$$

In accordance with (3), the pressure differs from the elastic potential only by sign:

$$q = -U = -\frac{c}{4r}\Big[1 + \frac{mbc}{2r}\Big(3 - 2\cos v\ln\frac{bc}{r}\Big)\Big]. \qquad (49)$$

The dependent components of stresses are determined by formulas (1) in accordance with (45), (47), and (49). At the site orthogonal to the contour circumference, the normal stress is expressed in terms of Cartesian stresses in the form $P^L_{vv} = P^L_{xx}\sin^2 v - 2P^L_{xy}\sin v\cos v + P^L_{yy}\cos^2 v$. From this relation, with allowance for stresses (1) expressed in terms of the independent stresses and pressure, we obtain

$$P^L_{vv} = -q^L + (P^L_{zx}\sin v - P^L_{zy}\cos v)^2/U'_L.$$

For the approximation linear in m, formulas (46)–(49) yield the relations

$$q^L = -\frac{c}{4r_0}\Big[1 + \frac{mbc}{2r_0}\Big(3 - 2\cos v\ln\frac{bc}{r_0}\Big)\Big], \qquad \frac{1}{U'_L} = -\frac{1}{2b}\Big(1 - m\frac{bc}{r_0}\Big),$$

$$(P^L_{zx}\sin v - P^L_{zy}\cos v)^2 = bc\,\frac{1-\cos v}{2r_0}\Big[1 + \frac{mbc}{r_0}\Big(1 + \ln\frac{bc}{r_0}\Big)\Big],$$

according to which the extensions of the hole contour for linear and quadratic elastic potentials have the form

$$P^{0L}_{vv} = \frac{c\cos v}{4r_0}, \qquad P^L_{vv} = \frac{c\cos v}{4r_0} + \frac{mbc^2}{8r_0^2}\Big(3 - 2\ln\frac{bc}{r_0}\Big).$$

It follows that, in the case considered, one part of the hole contour is extended and the other is compressed; taking into account potential nonlinearity under the natural condition (48) increases extension and decreases compression.

4. The solution of Eq. (36) in the stress plane that corresponds to the quadratic elastic potential and satisfies the boundary conditions in the physical plane can be found using the solution of the harmonic equation (32), which corresponds to the linear potential.

For the transformation of the independent variable

$$J_0 = \int \frac{dR_0}{R_0} = \ln R_0, \qquad \frac{\partial}{\partial J_0} = R_0\frac{\partial}{\partial R_0},$$

Eq. (37) becomes the Laplace equation

$$\frac{\partial^2 w_0}{\partial J_0^2} + \frac{\partial^2 w_0}{\partial V^2} = 0. \qquad (50)$$

The equation for displacements (36) can also be reduced to the Laplace equation by multiplying by $RU'/U_0'^2$ and setting

$$J = \int \frac{U_0'}{U'}\,\frac{dR}{R}, \qquad \frac{\partial}{\partial J} = \frac{RU'}{U_0'}\,\frac{\partial}{\partial R}, \qquad \frac{R^2U'}{U_0'}\,\frac{d}{dR}\,\frac{U'}{RU_0'} = -1. \tag{51}$$

As a result, we obtain

$$\frac{\partial^2 w}{\partial J^2} + \frac{\partial^2 w}{\partial V^2} = 0. \tag{52}$$

The last equality in (51) is the equation for the elastic potential

$$2\,\frac{U'}{RU_0'}\,\frac{d}{dR}\,\frac{U'}{RU_0'} = -\frac{2}{R^3}.$$

Integration of this equation yields $U'/U_0' = \sqrt{1+eR^2}$, $e = \text{const}$. For the condition $U' = U_0'$ to be satisfied for $m = 0$ and the potential have the form (11) for weak nonlinearity, the constant should be $e = 2m$. In this case, the potential and transformed coordinate in (51) become

$$U'/U_0' = \sqrt{1+2mR^2} \approx 1 + mR^2, \qquad J = \ln\,(R/\sqrt{1+mR^2}\,).$$

Thus, for weak nonlinearity of the potential, the equation for the displacement in the stress plane can be transformed to the harmonic equation (52).

Let the displacement

$$w = w_L \tag{53}$$

be specified on the cross-sectional contour L of the cylinder. First, we solve problem (32), (53) and find the displacement in the physical plane for the linear potential $w_0 = w_0(x, y)$. Then, using the relations between the coordinates $x = x(J_0, V)$ and $y = y(J_0, V)$, we transform it to the solution of Eq. (50) in the stress-plane variables $w_0 = w_0(J_0, V)$. Since Eqs. (52) and (50) are similar, the solution w of Eq. (52), which corresponds to the quadratic potential, can easily be obtained in the form $w = w_0(J, V)$ by substituting J for J_0 in the solution $w_0 = w_0(J_0, V)$. Finally, substituting the variables $J = J(x, y)$ and $V = V(x, y)$ into the resulting solution, we obtain the displacement in the physical plane $w(x, y) = w_0(J(x, y), V(x, y))$ and then, the corresponding stress field.

We now use the method considered above to solve the nonlinear boundary-value problem of the displacement in the exterior of an ellipse. In the case of a linear elastic potential, it is convenient to solve the problem by using conformal mapping.

Let the region S be the exterior of an ellipse with the center located at the coordinate origin and semiaxes k and l ($k > l$). The conformal mapping of the exterior of the ellipse onto the exterior of the unit circle $|\zeta| = 1$ is given by $z = p(\zeta + n/\zeta)$ and $\zeta = \rho e^{i\theta}$, where the parameters $p = (k+l)/2$ ($0 < p < \infty$) and $n = (k-l)/(k+l)$ ($0 < n < 1$) characterize the dimensions and shape of the ellipse, respectively. The ellipse becomes a circle for $n = 0$ and a cut for $n = 1$.

Let the boundary displacement have the form $w_L = c \ln r_L = c\,\mathrm{Re}\,(\ln z_L)$, where $c = \bar{c} = \text{const}$. In the complex variables z and $\bar{z}$, the harmonic equation (32) takes the form $\partial^2 w_0/\partial z\,\partial\bar{z} = 0$ and has the general solution $w_0 = \mathrm{Re}\,(\varphi(z))$, where $\varphi(z)$ is an arbitrary function. The boundary condition is satisfied if $\varphi(z) = c \ln z$. Hence, the displacement has the form $w_0 = (c/2)\ln(z\bar{z})$.

For the linear potential $U_0' = -2b$, formulas for stresses (28) can be written in the complex form

$$P^0_{zx} - iP^0_{zy} = 2b\Big(\frac{\partial w_0}{\partial x} - i\,\frac{\partial w_0}{\partial y}\Big) = 4b\,\frac{\partial w_0}{\partial z}.$$

Since

$$P^0_{zx} - iP^0_{zy} = R_0 e^{-iV} = e^{J_0 - iV}, \qquad \frac{\partial w_0}{\partial z} = \frac{c}{2z},$$

this equality and its inversion have the form

$$e^{J_0 - iV} = 2bc/z, \qquad z = 2bce^{-J_0 + iV}.$$

Substitution of the expressions for z and $\bar{z}$ into $w_0(z, \bar{z})$ yields the solution of Eq. (50): $w_0(J_0, V) = c(\ln(2bc) - J_0)$; replacement of the variable J_0 in this solution by $J = \ln\,(R/\sqrt{1+mR^2})$ gives the solution $w(R, V)$ of Eq. (36):

$$w(R,V)=w_0(R,V)=c\ln(2bc\sqrt{1+mR^2}/R). \tag{54}$$

Solution (54) and the potential U' from (11) correspond to functions (38) and transformation (39):

$$F(R,V)=-\frac{2bc}{R^2}\,\mathrm{e}^{iV},\quad G(R,V)=\frac{2ibc}{R}\,\mathrm{e}^{iV},\quad z=\frac{2bc}{R}\,\mathrm{e}^{iV},\quad \bar{z}=\frac{2bc}{R}\,\mathrm{e}^{-iV}.$$

Inverting the transformation, we obtain

$$R=2bc/r=2bc/\sqrt{x^2+y^2},\qquad \tan V=\tan v=y/x.$$

It follows that the Cartesian stress components in the physical plane have the form

$$P_{zx}=R\cos V=\frac{2bc}{r}\cos v=\frac{2bcx}{x^2+y^2},\quad P_{zy}=R\sin V=\frac{2bc}{r}\sin v=\frac{2bcy}{x^2+y^2}. \tag{55}$$

Stresses (55) vanish at infinity and take variable values at the boundary ellipse.

The conformal mapping implies the coordinate transformation

$$x=p(\rho+n/\rho)\cos\theta,\qquad y=p(\rho-n/\rho)\sin\theta. \tag{56}$$

Eliminating ρ or θ from these formulas, we obtain

$$\frac{x^2}{p^2(\rho+n/\rho)^2}+\frac{y^2}{p^2(\rho-n/\rho)^2}=1,\qquad \frac{x^2}{4np^2\cos^2\theta}-\frac{y^2}{4np^2\sin^2\theta}=1.$$

It follows from these relations that polar coordinates in the circle plane correspond to elliptic coordinates in the ellipse plane, the lines $\rho=\text{const}$ corresponding to ellipses and the lines $\theta=\text{const}$ to hyperbolas. At the boundary ellipse ($\rho=1$), stresses (55) are given by

$$P^L_{zx}=\frac{2bc(1+n)\cos\theta}{p(1+n^2+2n\cos 2\theta)},\qquad P^L_{zy}=\frac{2bc(1-n)\sin\theta}{p(1+n^2+2n\cos 2\theta)}. \tag{57}$$

In the variables ρ and θ, potentials (11) are written as

$$\begin{gathered}U=(bc^2/r^2)(1+2mb^2c^2/r^2),\qquad U'=-2b(1+4mb^2c^2/r^2),\\ r^2=(p^2/\rho^2)(\rho^4+n^2+2n\rho^2\cos 2\theta).\end{gathered} \tag{58}$$

The constant h determined by (4) can be calculated in the elliptic coordinates (56) as the limit of the mean value of the elastic potential in an elliptic ring that encloses the hole as the ring expands unlimitedly. Using the tables of integrals [7], we obtain

$$h=\lim_{\rho_*\to\infty}\frac{1}{S_*}\int_1^{\rho_*}d\rho\int_0^{2\pi}U(\rho,\theta)I(\rho,\theta)\,d\theta=-\frac{2bc^2}{p^2}\lim_{\rho_*\to\infty}\Big(\frac{\rho_*^2}{\rho_*^2-1}\,\frac{\ln\rho_*}{\rho_*^2+n}\Big)=0,$$

where

$$S_*=(\pi p^2/\rho_*^2)(\rho_*^2-1)(\rho_*^2+n),\qquad I=(p^2/\rho^3)(\rho^4+n^2-2n\rho^2\cos 2\theta).$$

Therefore, expressions for pressure (3) in the region and on the boundary are written as

$$\begin{gathered}q=-U=-(bc^2/r^2)(1+2mb^2c^2/r^2),\qquad r^2=(p^2/\rho^2)(\rho^4+n^2+2n\rho^2\cos 2\theta),\\ q_L=(bc^2/(p^2f))(1+2mb^2c^2/(p^2f)),\qquad f=1+n^2+2n\cos 2\theta.\end{gathered} \tag{59}$$

Formulas (55) and (57)–(59) determine the dependent stress components (1) and the contour load (6). Indeed, in accordance with (56) and (58), the elastic potentials and components of the normal have the following values on the contour:

$$U_L=\frac{bc^2}{fp^2}\Big(1+\frac{2mb^2c^2}{fp^2}\Big),\qquad \frac{1}{U'_L}=-\frac{1}{2b}\Big(1-\frac{4mb^2c^2}{fp^2}\Big),$$

$$n_x=-\frac{dy}{ds}=-\frac{1-n}{\sqrt{g}}\cos\theta,\quad n_y=\frac{dx}{ds}=-\frac{1+n}{\sqrt{g}}\sin\theta,\quad g=1+n^2-2n\cos 2\theta.$$

Using these relations, with allowance for (5) and (57), we obtain the contour stresses g_t and g_n and the contour load (6):

$$g_n = P_{zx}^L n_x + P_{zy}^L n_y = -2bc(1-n^2)/(fp\sqrt{g}), \qquad g_t = P_{zy}^L n_x - P_{zx}^L n_y = 4bcn\sin 2\theta/(fp\sqrt{g}),$$

$$p_t = \frac{4n(1-n^2)bc^2}{f^2gp^2}\Big(1 - \frac{4mb^2c^2}{fp^2}\Big)\sin 2\theta, \qquad p_b = -2bc\,\frac{1-n^2}{fp\sqrt{g}},$$

$$p_n = \frac{bc^2}{fp^2}\Big[1 - \frac{2(1-n^2)^2}{fg} + \frac{2mb^2c^2}{fp^2}\Big(1 + \frac{4(1-n^2)^2}{fg}\Big)\Big].$$

Extension of the hole contour is determined by the elliptic stress component $P_{\theta\theta}^L$ (normal stress at the site orthogonal to the elliptic boundary). Representing this component in terms of Cartesian components for transformation of coordinates (56) and expressing the Cartesian stress components in terms of independent stresses and pressure according to (1), we obtain

$$P_{\theta\theta}^L = -q_L + [P_{zx}^L(1+n)\sin\theta - P_{zy}^L(1-n)\cos\theta]^2/(gU_L'),$$

$$g = 1 + n^2 - 2n\cos 2\theta, \qquad 1/U_L' = -(1 - 4mb^2c^2/(p^2f))/(2b).$$

Substitution of stress (57) and pressure (59) into this equality yields

$$P_{\theta\theta}^L = P_{\theta\theta}^{0L} + mP_{\theta\theta}^{1L}, \quad P_{\theta\theta}^{0L} = \frac{bc^2}{p^2f}\Big(1 - \frac{8n^2}{fg}\sin^2 2\theta\Big), \quad P_{\theta\theta}^{1L} = \frac{2b^3c^4}{p^4f^2}\Big(1 + \frac{16n^2}{fg}\sin^2 2\theta\Big).$$

In the range of variation of the parameter n, the quantities f and g satisfy the inequalities

$$f \geqslant (1-n)^2 > 0, \quad g \geqslant (1-n)^2 > 0, \quad fg > 0 \qquad \text{for} \quad 0 < n < 1.$$

Using the relation $fg = (1-n^2)^2 + 4n^2\sin^2 2\theta$, we write $P_{\theta\theta}^{0L}$ in the form

$$P_{\theta\theta}^{0L} = \frac{bc^2}{p^2f}\,\frac{(1-n^2)^2 - 4n^2\sin^2 2\theta}{(1-n^2)^2 + 4n^2\sin^2 2\theta}.$$

It follows that, for the linear potential, the character of extension of the hole contour strongly depends on its shape:

$$P_{\theta\theta}^{0L} = \frac{bc^2}{p^2} > 0 \quad \text{for} \quad n = 0, \qquad P_{\theta\theta}^{0L} = -\frac{bc^2}{4p^2\cos^2\theta} < 0 \quad \text{for} \quad n = 1.$$

Therefore, by virtue of continuity of the stress, the contours of slightly extended (nearly circular) holes are extended, whereas the contours of strongly extended (close to a cut) holes are compressed. The additional stress associated with potential nonlinearity is positive: $P_{\theta\theta}^{1L} > 0$. Consequently, in this case as in the cases considered above, taking into account potential nonlinearity increases extension and decreases compression.

If the contour L with the equations $x = x(s)$ and $y = y(s)$ and the normal defined by $n_x = y'(s)$ and $n_y = -x'(s)$ (s is the arc length of L; the prime denotes the derivative with respect to s) is subjected to the admissible load $p_t(s)$, $p_n(s)$, and $p_b(s)$, relations (12), (5), and (11) determine the stresses $P_{zx}(s)$ and $P_{zy}(s)$ and the potential $U'(s)$. In this case, Eqs. (28) [compatible by virtue of the first equality in (5)] determine the displacement on L

$$w_L = w_0 + \frac{1}{2b}\int_0^s \frac{x'P_{zx} + y'P_{zy}}{1 + m(P_{zx}^2 + P_{zy}^2)}\,ds,$$

where w_0 is a specified constant. Thus, the problem is reduced to the problem considered above.

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